Seiberg-Witten invariants of rational homology 3-spheres

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Abstract

We prove that the Seiberg-Witten invariants of a rational homology sphere are determined by the Casson-Walker invariant and the Reidemeister torsion.

Introduction

In 1996 Meng and Taubes [11] have established a relationship between the Seiberg-Witten invariants of a (closed) 3-manifold with $b_1 \geq 0$. A bit later Turaev [21, 22] refined Meng-Taubes' result, essentially identifying these two invariants for such 3-manifolds. In [21] Turaev left open the question of establishing a connection between these two invariants in the remaining case, that of rational homology spheres.

Around the same time, Lim [8] succeeded in providing a combinatorial description of the Seiberg-Witten invariants of integral homology spheres. Namely, in this case they coincide with the Casson invariant. In [12] we investigated a special class of rational homology spheres, the lens spaces, and we proved that the Seiberg-Witten invariants of such spaces are equivalent to the Casson-Walker invariant and the Reidemeister torsion, and raised the question whether this is the case in general. Recently Marcolli and Wang [10] (see also the related work of Ozsváth-Zsabó [15]) have shown that the Seiberg-Witten invariants of a $\mathbb{Q}HS$ determine the Casson-Walker invariant. Additionally, they have proved a very general surgery formula involving the Seiberg-Witten invariants.

In the present paper we prove that for rational homology spheres we have

$$\text{SW} \Longleftrightarrow \text{modified Reidemeister torsion} \stackrel{def}{:=} \text{Casson-Walker} + \text{Reidemeister torsion}.$$

Our strategy is based on surgery formulæ for Seiberg-Witten invariants developed in [10], and surgery formulæ for the modified torsion, described in [22, 23].

Both the Seiberg-Witten invariant and the modified Reidemeister torsion can be thought of as $\mathbb{Q}$-valued functions on the first homology group H of a given rational homology sphere M. We denote by D_M the difference of these two functions. Proving the equality of these two invariants is equivalent to showing that $D_M \equiv 0$.

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We found it extremely convenient to work not with D_M but with its Fourier transform $\widehat{D}_M : H^\sharp \to \mathbb{C}$, where $H^\sharp$ is the dual of H. For example, Marcolli-Wang result [10] translates into $\widehat{D}_M(1) = 0$, for all rational homology spheres. Additionally, the true nature of the surgery formulæ is better displayed in the Fourier picture. To explain the gist of these formulæ consider a 3-manifold N with $b_1 = 1$ and boundary T^2. N can be thought of as the complement of a knot in a $\mathbb{Q}HS$. Pick two simple closed curves c_1, c_2 on ∂N with nontrivial intersection numbers with the longitude $\lambda \in H_1(\partial N, \mathbb{Z})$.

By Dehn surgery with c_i as attaching curves we obtain two rational homology spheres M_1, M_2 and two knots $K_i \hookrightarrow M_i$, $i = 0, 1$. Let $H_i := H_1(M_i, \mathbb{Z})$, $G := H_1(N, \partial N; \mathbb{Z})$. The knot K_i determines a subgroup $K_i^\perp \subset H_i^\sharp$, the characters vanishing on K_i. These subgroups are naturally isomorphic to G and thus we have a natural isomorphism

$$f : K_1^\perp \to K_2^\perp.$$

The surgery formulæ have the form

$$\langle \lambda, c_2 \rangle \widehat{D}_M(f^*\chi) = \langle \lambda, c_1 \rangle \widehat{D}_{M_2}(\chi) + |G|\mathcal{K}, \ \ \forall \chi \in K_2^\perp$$

where $\langle \bullet, \bullet \rangle$ denotes the intersection pairing on $H_1(\partial N, \mathbb{Z})$, and $\mathcal{K}$ is an universal correction term which depends only on the divisibility m_0 of the longitude, and the $SL_2(\mathbb{Z})$-orbit of the pair (c_1, c_2) with respect to the obvious action of this group on the space of pairs of primitive vectors in a 2-dimensional lattice. We will thus write $\mathcal{K}_{m_0;[c_1,c_2]}$, and call the triplet $(m_0; [c_1, c_2])$ the arithmetic type of the surgery. The results of [15] prove that

$$\mathcal{K}_{1;[c_1,c_2]} \equiv 0, \ \ \forall [c_1, c_2].$$

We call surgeries with $m_0 = 1$ primitive. The admissible surgeries have trivial correction term. We denote by $\mathfrak{X}$ the class of rational homology spheres M such that $\widehat{D}_M \equiv 0$. Both the family of admissible surgeries and the family $\mathfrak{X}$ are "time dependent" families, and during our proof we will gradually produce larger and larger classes of surgeries/ manifolds inside these families.

The class $\mathfrak{X}$ is closed under connected sums and certain primitive surgeries (see §4.1). Using this preliminary information we are able to show that all homology lens spaces belong to $\mathfrak{X}$. The proof uses Kirby calculus, and we learned it from Nikolai Saveliev. As a bonus, we can include many more arithmetic types of Dehn surgeries in the class of admissible surgeries.

Loosely speaking, the homology lens spaces have the simplest linking forms. We take this idea seriously, and we define an appropriate notion of complexity of a linking form. The proof then proceeds by induction, including in $\mathfrak{X}$ manifolds of larger and larger complexity. This process also increases the class of admissible surgeries, which can be used at the various inductive steps. Such a proof is feasible if we can produce a large supply of complexity reducing Dehn surgeries. Fortunately, this is can be seen using elementary arithmetic.

Our proof also shows that the invariant introduced by Ozsváth and Szabó in [15] also coincides with Casson-Walker + Reidemeister torsion, thus answering a question raised in that paper. Moreover our result establishes connections between the Kreck-Stolz invariant [6], the linking form, and an invariant we introduced in [14].

Acknowledgments I learned about the Reidemeister torsion from Frank Connolly. He explained the basics to me, and over countless hours of discussions he listened and answered patiently to many of my questions and my half-developed ideas. These discussions always enriched my knowledge, and made me aware of many of the subtleties of this subject.

I am grateful to Nikolai Saveliev for his patience and expertise in answering the questions I had while working on this project. In particular, his proof of Lemma 4.5 got me "out of a hole". I want to thank my colleagues Stephan Stolz and Larry Taylor for many illuminating discussions.

Basic Notations and Terminology A closed, compact, oriented 3-manifold will be denoted by M. We will set $H = H_1(M,\mathbb{Z}) \cong H^2(M,\mathbb{Z})$, and we will denote the group operation multiplicatively. We denote by $T(H)$ the torsion part of H and by $T_2(H)$ the 2-torsion part

$$T_2(H) := \bigl\{h \in T(H);\;\; h^2 = 1\bigr\}.$$

We set $\Theta = \Theta_M := \sum_{h\in T(H)} h \in \mathbb{Z}[H]$. For any $P = \sum_{h\in H} P_h h \in \mathbb{Z}[H]$ we set

$$\bar{P} = \sum_{h\in H} P_h h^{-1}.$$

The letter N will be reserved for compact, oriented three manifolds with boundary $\partial N \cong T^2$ such that $b_1(N) = 1$. Equivalently, N can be viewed as the complement of a knot in a rational homology sphere. We set $G = H_1(N, \partial N) \cong H^2(N,\mathbb{Z})$.

We will denote by $Spin^c(M)$ the space of isomorphism classes of *spin*^{c} structures on M. We will denote a generic *spin*^{c} structure by σ. $Spin^c(M)$ is a H-torsor, and we will denote by

$$Spin^c(M) \times H \ni (\sigma, h) \mapsto h\sigma$$

the natural action of H on $Spin^c(M)$. The natural involution on $Spin^c(M)$ will be denoted by $\sigma \mapsto \bar{\sigma}$. The complex line bundle associated to σ will be denoted by $\det(\sigma)$. We can identify $\det(\sigma)$ via the first Chern class with an element in H. Note that

$$\det(h\sigma) = h^2 \det(\sigma).$$

We will denote by $Spin(M)$ the space of isomorphism classes of *spin*-structures on M. A generic *spin* structure will be denoted by ϵ. $Spin(M)$ is naturally a $T_2(H)$-torsor. We use the same notation to denote the action of $T_2(H)$ on $Spin(M)$. Every *spin* structure ϵ induces a canonical *spin*^{c}-structure $\sigma(\epsilon)$. Moreover

$$\sigma(\epsilon) = \overline{\sigma(\epsilon)}, \;\; \sigma(h\epsilon) = h\sigma(\epsilon), \;\; \forall h \in T_2(H).$$

For any Abelian group A we will denote by $A^\sharp$ its dual, $A^\sharp = \mathrm{Hom}(A, S^1)$. Finally for every $\chi \in H^\sharp$, and any $P = \sum_{h\in H} P_h h \in \mathbb{C}[H]$ we set

$$\hat{P}(\chi) := \sum_{h\in H} P_h \chi(h) \in \mathbb{C}.$$

The function $H^\sharp \ni \chi \mapsto \hat{P}(\chi)$ is essentially the Fourier transform of P. Note that

$$\hat{P}(1) := \sum_{h \in H} P_h.$$

Moreover

$$\hat{\Theta}_M(1) = |T(H)|, \;\; \hat{\Theta}_M(\chi) = 0, \;\text{ if }\; \chi \neq 1, \;\text{ and }\; \exists m > 1 \;\; \chi(H) \subset U_m.$$

For every positive integer m we denote by $U_m \subset S^1$ the multiplicative group of m-th roots of 1.

Contents

1 The modified Seiberg-Witten invariants of 3-manifolds

We want to present in a form suitable for our goals, some basic structural facts concerning the Seiberg-Witten invariants of 3-manifolds. For more details we refer to [2, 11, 9].

§1.1 The case $b_1 > 1$. Suppose $b_1(M) > 1$. Fix an orientation on $H \otimes \mathbb{R}$. The Seiberg-Witten invariant of M is a function

$$\mathbf{sw}_M : Spin^c(M) \to \mathbb{Z}.$$

$\mathbf{sw}_M(\sigma)$ is a signed count of σ-monopoles, objects determined by additional geometric data on M, Riemann metric g and a closed 2-form η. The chosen orientation on $H \otimes \mathbb{R}$ associates a sign to each monopole, and the signed count is independent of g and η. The Seiberg-Witten invariant has the following properties.

- $\mathbf{sw}_M(\sigma) = 0$ for all but finitely many σ's.
- $\mathbf{sw}_M(\sigma) = \mathbf{sw}_M(\bar{\sigma})$, $\forall \sigma$.

For every σ we can form the element

$$\mathbf{SW}_{M,\sigma} \in \mathbb{Z}[H], \;\; \mathbf{SW}_{M,\sigma} = \sum_{h \in H} \mathbf{sw}_M(h^{-1}\sigma)h.$$

Note that for every $h_0 \in H$ we have

$$\mathbf{SW}_{M,h_0\sigma} = h_0 \mathbf{SW}_{M,\sigma}.$$

Moreover

$$\mathbf{SW}_{M,\sigma} = \det(\sigma)\mathbf{SW}_{M,\bar{\sigma}} = \det(\sigma)\overline{\mathbf{SW}}_{M,\sigma}.$$

In particular, for any *spin* structure ϵ we have $\det(\sigma(\epsilon)) = 1$ so that

$$\mathbf{SW}_{M,\sigma(\epsilon)} = \overline{\mathbf{SW}}_{M,\sigma(\epsilon)}.$$

For simplicity we set $\mathbf{SW}_{M,\epsilon} := \mathbf{SW}_{M,\sigma(\epsilon)}$, $\forall \epsilon$.

§1.2 The case $b_1 = 1$. Suppose $b_1(M) = 1$, and fix an orientation on $H \otimes \mathbb{R}$. In this case, a choice of orientation is determined by fixing an isomorphism $H \otimes \mathbb{R} \to \mathbb{R}$. To describe the Seiberg-Witten invariant of M we need to recall the rudiments of its construction. theory. Fix a metric g. The chosen orientation on $H \otimes \mathbb{R}$ defines a harmonic 1-form ω_g such that ω_g induces the chosen orientation on $H \otimes \mathbb{R}$, and $\|\omega_g\|_{L^2(g)} = 1$. By rescaling the metric we can assume that ω_g also generates the image of H in $H \otimes \mathbb{R}$.

Note that the chosen orientation produces an isomorphism $H/(T(H) \to \mathbb{Z}$, and thus a map

$$\deg : H \to H/T(H) = \mathbb{Z}.$$

For $\sigma \in Spin^c(M)$ denote by $\mathcal{P}_\sigma(g)$ the space of closed 2-forms such that

$$w(\sigma, \eta) := \int_M \omega_g \wedge \eta - 2\pi c_1(\det \sigma) \neq 0.$$

It is decomposed into two chambers

$$\mathcal{P}^{\pm}_\sigma(g) = \Big\{ \eta \in \mathcal{P}_\sigma(g); \;\; \pm w(\sigma, \eta) > 0 \Big\}.$$

For $\eta \in \mathcal{P}_\sigma(g)$ we denote by $\mathbf{sw}^{\pm}_M(\sigma, \eta)$ the signed count of (σ, g, η)-monopoles. It is known that

$$\mathbf{sw}_M(\sigma, \eta) = \mathbf{sw}_M(\bar{\sigma}, \eta)$$

$\mathbf{sw}_M(\sigma, \eta) = 0$ for all but finitely many σ's, and

$$\mathbf{sw}_M(\sigma, \eta_1) = \mathbf{sw}_M(\sigma, \eta_2), \ \text{ if } \ w(\sigma, \eta_1) \cdot w(\sigma, \eta_2) > 0.$$

We set

$$\mathbf{sw}^{\pm}_M(\sigma) := \mathbf{sw}_M(\sigma, \eta), \ \ \pm w(\sigma, \eta) > 0.$$

The wall crossing formula (see [9]) states that

$$\mathbf{sw}^{+}_M(\sigma) - \mathbf{sw}^{-}_M(\sigma) = \frac{1}{2}\deg(\det \sigma).$$

Set

$$\mathbf{SW}_{M,\sigma,\eta} = \sum_{h \in H} \mathbf{sw}_M(\sigma, \eta) \in \mathbb{Z}[H], \ \ \mathbf{SW}_{M,\sigma} = \sum_{h \in H} \mathbf{sw}^{+}_M(h^{-1}\sigma)h \in \mathbb{Z}[[H]].$$

Suppose we pick $\sigma = \sigma(\epsilon)$ and $\eta = \eta_0$ such that $\int_M \omega \wedge \eta_0$ is a very small positive number. Fix $T \in H$ such that $\deg(T) = 1$. We can rephrase the wall crossing formula in the more compact form

$$\mathbf{SW}_{M,\sigma(\epsilon)} = \mathbf{SW}_{M,\sigma(\epsilon),\eta_0} + \frac{\Theta_M T}{(1-T)^2}.$$

We set $W_M := \frac{\Theta_M T}{(1-T)^2} \in \mathbb{Z}[[H]]$, and we will refer it as *wall-crossing correction*[1] of M. Observe that

$$\mathbf{SW}^0_{M,\sigma(\epsilon)} := \mathbf{SW}_{M,\sigma(\varepsilon)} - W_M = \mathbf{SW}_{M,\sigma(\epsilon),\eta_0} \in \mathbb{Z}[H]$$

satisfies the symmetry condition

$$\mathbf{SW}^0_{M,\sigma(\epsilon)} = \overline{\mathbf{SW}}^0_{M,\sigma(\epsilon)},$$

and

$$\mathbf{SW}^0_{M,\sigma(h_0\epsilon)} = h_0 \mathbf{SW}^0_{M,\sigma(\epsilon)}, \ \ \forall h_0 \in T_2(H).$$

We will refer to $\mathbf{SW}^0_{M,\sigma(\epsilon)}$ as *the modified Seiberg-Witten invariant of M.*

[1] Observe that the wall-crossing correction $\Theta_M T(1-T)^{-2} \in \mathbb{Z}[[H]]$ is independent of the choice of T in H such that $\deg T = 1$.

§1.3 The case $b_1 = 0$. Suppose now that $b_1(M) = 0$, i.e. M is a rational homology sphere. Fix $\sigma \in Spin^c(M)$. In this case the signed count of (σ, g, η)-monopoles depends on $(g, \eta$ in a more complicated way. To produce a topological invariant we need to add a correction to this count. For simplicity, we describe this correction only when $\eta = 0$.

Denote by $\mathbb{S}_\sigma$ the bundle of complex spinors determined by σ. The line bundle $\det \sigma = \det \mathbb{S}_\sigma$ admits an unique equivalence class of flat connections. Pick one such flat connection A_σ and denote by $\mathfrak{D}_{A_\sigma}$ Dirac operator on $\mathbb{S}_\sigma$ determined by the twisting connection σ. We denote its eta invariant by $\eta_{dir}(g, \sigma)$. Also, denote by $\eta_{sign}(g)$ the eta invariant of the odd signature operator determined by g. Finally define the Kreck-Stolz invariant of (g, σ) by

$$KS(g, \sigma) = 4\eta_{dir}(g, \sigma) + \eta_{sign}(g).$$

Define *the modified Seiberg-Witten invariant* of (M, σ) by

$$\mathbf{sw}^0_M(\sigma) = \frac{1}{8} KS(g, \sigma) + \mathbf{sw}_M(\sigma) \in \mathbb{Q}.$$

As shown in [9], the above quantity is independent of the metric, and it is a topological invariant. Set

$$\mathbf{SW}^0_{M,\sigma} := \sum_{h \in H} \mathbf{sw}_M(h^{-1}\sigma) h \in \mathbb{Q}[H].$$

If $\sigma = \sigma(\epsilon)$ we have

$$\mathbf{SW}^0_{M,\sigma(\epsilon)} = \overline{\mathbf{SW}}^0_{M,\sigma(\epsilon)}.$$

§1.4 Summary Let us coherently organize the facts explained so far. We say that M is *homologically oriented* if $H \otimes \mathbb{R}$ is oriented. The modified Seiberg-Witten invariant associates to each closed, compact, homologically oriented 3-manifold M, and each $\epsilon \in Spin(M)$ a "Laurent polynomial" $\mathbf{SW}^0_{M,\epsilon} \in \mathbb{Q}[H]$ with the following properties.

$$\mathbf{SW}^0_{M,\epsilon} \in \mathbb{Z}[H], \text{ if } b_1(M) > 0, \tag{1.1}$$

$$\mathbf{SW}^0_{M,\epsilon} = \overline{\mathbf{SW}}^0_{M,\epsilon}, \tag{1.2}$$

and

$$\mathbf{SW}^0_{M,h_0\epsilon} = h_0 \mathbf{SW}^0_{M,\epsilon}, \quad \forall h_0 \in T_2(H). \tag{1.3}$$

2 The modified Reidemeister-Turaev torsion of 3-manifolds

In this section we survey the results of V. Turaev [18, 19, 20, 21, 22] in a language appropriate to our goals.

§2.1 Turaev's refined torsion The Reidemeister-Turaev torsion associates to each of a homologically oriented 3-manifold M, and each *spin*^{c} structure σ on M a "formal power series" $\mathcal{T}_{M,\sigma} \in \mathbb{Q}[[H]]$ with the following properties.

$$\mathcal{T}_{M,\sigma} \in \mathbb{Z}[[H]], \;\; b_1(M) > 1$$

$$(1-T)^2\mathcal{T}_{M,\sigma} \in \mathbb{Z}[H], \;\; b_1(M) = 1, \;\; \deg T = 1$$

$$\mathcal{T}_{M,\sigma} \in \mathbb{Q}[H], \;\; , \;\; \mathcal{T}_{M,\sigma}(1) = 0, \;\; b_1(M) = 0.$$

Moreover

$$\mathcal{T}_{M,h_0\sigma} = h_0\mathcal{T}_{M,\sigma}, \;\; \forall h_0 \in H,$$

and

$$\mathcal{T}_{M,\sigma} = \det(\sigma)\overline{\mathcal{T}}_{M,\sigma}.$$

For $\epsilon \in Spin(M)$ set $\mathcal{T}_{M,\epsilon} = \mathcal{T}_{M,\sigma(\epsilon)}$. It follows that

$$\mathcal{T}_{M,\epsilon} = \overline{\mathcal{T}}_{M,\epsilon}.$$

Using [20, 21] and [22, Appendix 3] we deduce that when $b_1(M) = 1$ we have

$$\mathcal{T}^0_{M,\epsilon} := \mathcal{T}_{M,\epsilon} - W_M \in \mathbb{Z}[H],$$

and moreover

$$\mathcal{T}^0_{M,\epsilon} = \overline{\mathcal{T}}^0_{M,\epsilon}.$$

When $b_1(M) = 0$ denote by $CW(M)$ the Casson-Walker invariant of M and define

$$\mathcal{T}^0_{M,\epsilon} = \mathcal{T}_{M,\epsilon} + \frac{1}{2}CW(M)\Theta_M.$$

Observe that $\mathcal{T}^0_{M,\epsilon}(1) = \frac{1}{2}|H|CW(M)=$ Lescop invariant of M (see [7, p. 80]).

We we will refer to the quantities $\mathcal{T}^0_{M,\epsilon}$ for $b_1(M) = 0, 1$ the *modified Reidemeister-Turaev torsion of* M. For uniformity, we set $\mathcal{T}^0_M = \mathcal{T}_M$ when $b_1(M) > 1$. Summarizing, we conclude that the modified Reidemeister-Turaev torsion associates to each homologically oriented 3-manifold M, and to each *spin* structure ϵ on M a "Laurent polynomial" $\mathcal{T}^0_{M,\epsilon} \in \mathbb{Q}[H]$ with the following properties.

$$\mathcal{T}^0_{M,\epsilon} \in \mathbb{Z}[H], \;\; \text{if} \;\; b_1(M) > 0, \tag{2.1}$$

$$\mathcal{T}^0_{M,\epsilon} = \overline{\mathcal{T}}^0_{M,\epsilon}, \tag{2.2}$$

and

$$\mathcal{T}^0_{M,h_0\epsilon} = h_0\mathcal{T}^0_{M,\epsilon}, \;\; \forall h_0 \in T_2(H). \tag{2.3}$$

§2.2 Relations between the torsion and the Seiberg-Witten invariant The Seiberg-Witten invariant and the modified Reidemeister torsion are related. More precisely we have the following result.

Theorem 2.1. *(a)* $\mathbf{SW}^0_{M,\epsilon} = \mathcal{T}^0_{M,\epsilon}$ *if* $b_1(M) > 0$*; see* **[11, 21, 22]**.
(b)$\mathbf{SW}^0_M(1) = \mathcal{T}^0_M(1)$ *if* $b_1(M) = 0$*; see* **[2, 8]**.
(c)$\mathbf{SW}^0_M = \mathcal{T}^0_M$ *if* M *is a lens space; see* **[12]**.

Part (c) of the above theorem can be slightly strengthened to

$$\mathbf{SW}^0_M = \mathcal{T}^0_M, \text{ if } M \text{ is a connected sum of lens spaces.} \tag{2.4}$$

This equality follows from the vanishing of the torsion under connected sums, the additivity of the Casson-Walker invariant, and the additivity of the Kreck-Stolz invariant. (This follows from the very general surgery results for eta invariants in [5].)

Later on we will need the following consequence of Theorem 2.1 (a).

Proposition 2.2. *If* M *is a homologically oriented* 3*-manifold such that* $b_1(M) = 1$ *then*

$$\widehat{\mathcal{T}}^0_M(1) = \widehat{\mathbf{SW}}^0_M(1) = \frac{1}{2}\Delta''_M(1),$$

where $\Delta_M \in \mathbb{Z}[[T^{1/2}, T^{-1/2}]]$ *denotes the symmetrized Alexander polynomial of* M *normalized such that* $\Delta_M(1) = |T(H)|$.

Proof The projection $H \to H/T(H) = \mathbb{Z}$ induces a morphism

$$\mathfrak{aug} : \mathbb{Z}[[H]] \to \mathbb{Z}[t, t^{-1}]]$$

called augmentation. Fix $T \in H$ such that $\deg T = 1$. The symmetrized Alexander polynomial Δ_M is uniquely determined by the condition

$$\mathfrak{aug}\mathcal{T}_{M,\epsilon} = T^{k/2}\frac{\Delta_M(T)}{(1-T)^2}$$

for some $k \in \mathbb{Z}$. Using Theorem 2.1(a) we deduce

$$T^{k/2}\frac{\Delta_M(T)}{(1-T)^2} = \mathfrak{aug}\mathbf{SW}_M = \mathfrak{aug}\mathbf{SW}^0_M + \mathfrak{aug}(\Theta_M)\frac{T}{(1-T)^2} = \mathfrak{aug}\mathbf{SW}^0_M + |H|\frac{T}{(1-T)^2}.$$

We conclude that

$$T^{k/2-1}\Delta_M(T) = (T - 2 + T^{-1})\mathfrak{aug}\mathbf{SW}^0_M(T) + |H|.$$

The symmetry of $\mathbf{SW}^0$ implies $\mathbf{SW}^0_M(T) = \mathbf{SW}^0_M(T^{-1})$, and since Δ_M satisfies a similar symmetry we conclude $k/2 - 1 = 0$. Hence

$$\Delta_M(T) = (T - 2 + T^{-1})\mathfrak{aug}\mathbf{SW}^0_M(T) + |H|.$$

Differentiating the above equality twice at $T = 1$ we deduce

$$\Delta''_M(1) = 2\widehat{\mathfrak{aug}\,\mathbf{SW}}_M(1) = 2\widehat{\mathbf{SW}}^0(1). \quad \blacksquare$$

Remark 2.3. Observe a nice "accident". Suppose M is as in Proposition 2.2. Then

$$W_M = \Theta_M \sum_{n\geq 1} nT^n.$$

Formally

$$\widehat{W}_M(1) = \widehat{\Theta}_M(1) \sum_{n\geq 1} n = |H| \sum_{n\geq 1} n = |H|\zeta(-1) = -\frac{1}{12}|H|,$$

where $\zeta(s)$ denotes Riemann's zeta function. In particular

$$\widehat{\mathbf{SW}}_M(1) = \widehat{\mathbf{SW}}^0_M(1) + \widehat{W}_M(1) = \frac{1}{2}\Delta''_M(1) - \frac{1}{12}|H|.$$

The expression in the right-hand-side is precisely the Lescop invariant of M.

We can now state the main result of this paper.

Theorem 2.4.

$$\mathbf{SW}^0_M = \mathcal{T}^0_M$$

for any, closed, compact, oriented 3*-manifold* M*.*

3 Surgery formulæ

§3.1 Dehn surgery We want to survey a few basic facts concerning Dehn surgery. For more details and examples we refer to [13].

Consider a 3-manifold N as in the introduction, i.e. $b_1(N) = 1$, $\partial N \cong T^2$, and set $G := H_1(N, \partial N; \mathbb{Z})$. We orient ∂N as boundary of N using the outer-normal first convention. Denote by $\mathbf{j}$ the inclusion induced morphism

$$\mathbf{j} : H_1(\partial N, \mathbb{Z}) \to H_1(N, \mathbb{Z}).$$

The kernel of $\mathbf{j}$ is a rank one Abelian group. We can select a generator λ of $\ker \mathbf{j}$ by specifying an orientation on $H^1(N, \mathbb{Z}) \cong H_2(N, \partial N; \mathbb{Z})$. We can write $\lambda = m_0\lambda_0$ where $m_0 > 0$ and $\lambda_0 \in H_1(\partial N, \mathbb{Z})$ is a primitive class. λ is called the longitude of N and m_0 is called the *divisibility* of N.Fix $\mu_0 \in H_1(\partial N, \mathbb{Z})$ such that $\lambda_0 \cdot \mu_0 = 1$, where the dot denotes the intersection pairing on $H_1(\partial N, \mathbb{Z})$.

Denote by X the solid torus $S^1 \times D^2$, so that $\partial X = T^2$ Set $\ell_0 = S^1 \times \{\mathbf{pt}\}$ and $\mathbf{m}_0 = \{\mathbf{pt}\} \times \partial D^2$. We regard ℓ_0 and $\mathbf{m}_0$ as elements in $H_1(\partial X, \mathbb{Z})$. They satisfy $\mathbf{m}_0 \cdot \ell_0 = 1$. Fix an orientation reversing diffeomorphism $\Gamma : \partial X \to \partial N$ such that

$$\Gamma_*(\mathbf{m}_0) = \mu_0, \;\; \Gamma_*(\ell_0) = \lambda_0.$$

Every $\varphi \in SL_2(\mathbb{Z})$ determines an isotopy class of orientation preserving diffeomorphisms of T^2. We can for a closed 3-manifold

$$M_\varphi := X \coprod_{\Gamma\circ\varphi:\partial X \to \partial N} N.$$

We say that M_φ is obtained by *Dehn surgery* with gluing map φ. The integer m_0 is called the divisibility of the surgery. The manifold M_φ is uniquely determined up to a diffeomorphism by the attaching curve $c = \Gamma \circ \varphi(\mathbf{m}_0$. We can write $c = c_{p/q} := p\mu_0 + q\lambda_0$, $(p, q) = 1$. The diffeomorphism type of M_φ is uniquely determined by the ration p/q. Instead of M_φ we will write $M_{p/q}$. We set $H_{p/q} := H_1(M_{p/q}, \mathbb{Z})$. The core of the solid torus determines an element $K_{p/q} \in H_{p/q}$.

We want to point out that the integer q depends on the choice of μ_0 while p is invariantly determined by the equality

$$p := \lambda_0 \cdot c.$$

We we refer to p as the *multiplicity* of the surgery.

The group $H_{p/q}$ is determined from the short exact sequence

$$0 \to \langle \mathbf{j}c_{p/q} \rangle \to H_1(N, \mathbb{Z}) \to H_{p/q} \to 0.$$

We also have canonical isomorphisms

$$\Phi_{p/q} : G \to H_{p/q}/\langle K_{p/q} \rangle =: R_{p/q}.$$

We obtain a natural projection $\pi_{p/q} : H_{p/q} \to G$.

The long exact sequence of the pair $(N, \partial N)$ implies

$$G = H_1(N, \mathbb{Z})/\mathbf{j}H_1(\partial N, \mathbb{Z}).$$

We deduce the following result.

Lemma 3.1. *The characters of G are precisely the characters of $H_1(N, \mathbb{Z})$ which vanish on $\mathbf{j}H_1(\partial N, \mathbb{Z})$. Also, we can think of the characters of G as characters χ of $H_{p/q}$ such that $\chi(K_{p/q}) = 1$.*

When $p \neq 0$, $H_{p/q}$ is a finite Abelian group and

$$|H_{p/q}| = pm_0|G|.$$

In this case, we denote by $\mathbf{lk}_{p/q}$ the linking form of $M_{p/q}$.

Observe that $b_1(M_{0/1}) = 1$. $K_{0/1}$ can be written as $m_0 h$ where $h \in H_{0/1}$ generates the free part of $H_{0/1}$. $M_{0/1}$ carries a natural homology orientation, induced from the orientation of $H^1(N, \mathbb{Z})$ and $H^1(X, \mathbb{Z})$ (see [22] for more details on this rather painful issue). Fix $T \in H_{0/1}$ such that $\deg(T) = 1$, and $K_{0/1} = m_0 T$. There exists $\chi_0 \in H^\sharp_{0/1}$ uniquely determined by the requirements

$$\chi_0(T) = \rho, \;\; \chi_0|_{T(H_{0/1})} = 1,$$

where ρ is a primitive m_0-th root of 1. According to Lemma 3.1 we can think of χ_0 as a character of G.[2]

[2]The Universal Coefficients Theorem and the Poincaré duality identifies $G^\sharp = H_1(N, \partial N; \mathbb{Z})^\sharp$ with the torsion subgroup of $H_1(N, \mathbb{Z})$. Via this identification we have $\chi_0 = \mathbf{j}\lambda_0 \in H_1(N, \mathbb{Z})$.

§3.2 Surgery formula for the modified Seiberg-Witten invariant We can now state the main surgery formula for the modified Seiberg-Witten invariant. Let N, $M_{p/q}$ etc. be as above. Fix a *spin* structure ϵ on N. It induces *spin*-structures $\epsilon_{p/q}$ on $M_{p,q}$. For $h \in H_{p,q}$ We will write $\mathbf{sw}^0_{p/q}(h)$ for $\mathbf{sw}^0_{M_{p/q}}(h^{-1}\epsilon_{p/q})$.

Theorem 3.2. ([10, Marcolli-Wang], [15, Ozsváth-Szabó]) *For every p, q there exists*

$$f_{p,q,m_0} : U_{m_0} \to \mathbb{Q}$$

which depends only on p, q, m_0 but not on N such that for every $g \in G$ we have

$$\sum_{\pi_{p/q}(h)=g} \mathbf{sw}^0_{p/q}(h) = p \sum_{\pi_{1/0}(h)=g} \mathbf{sw}^0_{1/0}(h) + q \sum_{\pi_{0/1}(h)=g} \mathbf{sw}^0(h) + f_{p,q,m_0}(\chi_0(g)). \qquad (\dagger_g)$$

To get more information out of this formula we will take a partial Fourier transform. Let $\chi \in G^\sharp$. Using Lemma 3.1 we can identify χ with a character of $H_{p/q}$ with the property that $\chi(h) = \chi(h')$ whenever $\pi_{p/q}(h) = \pi_{p/q}$. If we multiply $(\dagger_g)$ by $\chi(g)$ and then we sum over $g \in G$ we deduce

$$\widehat{\mathbf{SW}}^0_{p/q}(\chi) = p\widehat{\mathbf{SW}}^0_{1/0}(\chi) + q\widehat{\mathbf{SW}}^0_{0/1}(\chi) + \sum_{g\in G} f_{p,q,m_0}(\chi_0(g))\chi(g).$$

To gain further insight we need to simplify the sum on the right hand side. We have

$$\sum_{g\in G} f_{p,q,m_0}(\chi_0(g))\chi(g) = \sum_{\rho\in U_{m_0}} \Big(\sum_{\chi_0(g)=\rho} f_{p,q,m_0}(\rho) \Big)\chi(g) = \sum_{\rho\in U_{m_0}} \Big(\sum_{\chi_0(g)=\rho} \chi(g) \Big) f_{p,q,m_0}(\rho)$$

Observe that if $\chi \not\equiv 1$ on $\ker \chi_0$ then

$$\sum_{\chi_0(g)=\rho} \chi(g) = 0, \;\; \forall \rho \in U_{m_0}.$$

If $\chi \equiv 1$ on $\ker \chi_0$ then there exists $j \in \mathbb{Z}$ such that $\chi = \chi_0^j$ and

$$\sum_{\chi_0(g)=\rho} \chi(g) = |\ker \chi_0|\rho^j = \frac{|G|}{m_0}\rho^j.$$

Denote by F_{p,q,m_0} the function

$$F_{p,q,m_0} : \mathbb{Z}/m_0\mathbb{Z} \to \mathbb{C}, \;\; F_{p,q,m_0}(j \mod \mathbb{Z}) = \frac{1}{m_0} \sum_{\rho\in U_{m_0}} f_{p,q,m_0}(\rho)\rho^j.$$

F_{p,q,m_0} is precisely the Fourier transform of $\frac{1}{m_0} f_{p,q,m_0}$. We deduce

$$\widehat{\mathbf{SW}}^0_{p/q}(\chi) = p\widehat{\mathbf{SW}}^0_{1/0}(\chi) + q\widehat{\mathbf{SW}}^0_{0/1}(\chi) + |G| \begin{cases} F_{p,q,m_0}(j) & \text{if} \quad \chi = \chi_0^j \\ 0 & \text{otherwise} \end{cases}. \qquad (3.1)$$

§3.3 Surgery formula for the modified Reidemeister-Turaev torsion The modified Reidemeister-Turaev torsion satisfies a surgery formula very similar in spirit to (3.1). We first need to survey a few algebraic facts in the special setting of the surgery formula. For details and proofs we refer to [13, 18].

For any finite Abelian group G we set

$$\mathbb{Q}[G]_0 = \Big\{ P \in \mathbb{Q}[G]; \;\; P(1) = 0 \Big\}.$$

Consider a rank 1 Abelian group $A = \mathbb{Z} \oplus Tors$, C a finite cyclic group of order m, and $\varphi : A \to C$ a surjective morphism. Fix a generator $u \in A$ of $\mathbb{Z} \subset A$, and let

$$\mathbb{Z}[[A]]_+ := \mathbb{Z}[A] + \Theta_A \mathbb{Z}[u, u^{-1}, (1-u)^{-1}].$$

(We refer to [18, 22] for an invariant definition of $\mathbb{Z}[[A]]_+$, which does not rely on the non-canonical decomposition $A =$ free part $\oplus$ torsion.) The morphism φ induces a morphism

$$\varphi_\sharp : \mathbb{Z}[[A]]_+ \to \mathbb{Z}[C]_0.$$

Its definition is best expressed in terms of Fourier transforms. Think of an element $f \in \mathbb{Z}[[A]]_+$ as a function $f : A \to \mathbb{Z}$. As such, it has a Fourier transform

$$\hat{f} : A^\sharp \to \mathbb{C}, \;\; \hat{f}(\chi) = \sum_{a \in h} f(a)\chi(h).$$

The above infinite sum may not be convergent for all χ, but the Fourier transform still makes sense as a distribution on the compact Lie group $A^\sharp$. There are finitely many characters χ for which the series is not convergent, and they are characterized by the condition $\chi(u) = 1$. We will call such characters *singular* and the other regular. We denote by $A^\sharp_{reg}$ the set of regular characters, and set $A^\sharp_{sing} := A^\sharp \setminus A^\sharp_{reg}$. The morphism φ induces by duality an inclusion

$$\varphi^\sharp : C^\sharp \hookrightarrow A^\sharp.$$

The Fourier transform of $\varphi_\sharp f$ is a function $\widehat{\varphi_\sharp f} : C^\sharp \to \mathbb{C}$. Let $\chi \in C^\sharp$. In [13] we have shown how to compute the value $\widehat{(\varphi_\sharp f)}(\chi) \in \mathbb{C}$. More precisely we have shown the following.

- If $f \in \mathbb{Z}[A]$, so that f has finite support as a function $A \to \mathbb{Z}$, we have

$$\widehat{(\varphi_\sharp f)}(\chi) = \left\{ \begin{array}{ccc} \hat{f}(\varphi^\sharp \chi) & \text{if} & \chi \neq 1 \\ 0 & \text{if} & \chi = 1 \end{array} \right. . \tag{3.2}$$

- If $f = \Theta_A (1-u)^{-1}$ then

$$\widehat{(\varphi_\sharp f)}(\chi) = \left\{ \begin{array}{ccc} \widehat{\Theta}_A(\varphi^\sharp \chi)\big(1 - \varphi^\sharp \chi(T)\big)^{-1} & \text{if} & \varphi^\sharp \chi \in A^\sharp_{reg} \\ & & \\ 0 & \text{if} & \varphi^\sharp \chi \in A^\sharp_{sing} \end{array} \right. . \tag{3.3}$$

Suppose now that $\chi : A \to U_m$ is a surjective character, and $f \in \mathbb{Z}[[A]]_+$. The identity function $\iota_m : U_m \to U_m$ is a character of U_m and $\chi^\sharp(\iota_m) = \chi$. We get an element $\chi_\sharp f \in \mathbb{Z}[U_m]_0$. If $f \in \mathbb{Z}[A]$ then

$$(\chi_\sharp f)(\iota_m) = \begin{cases} \hat{f}(\chi) & \text{if} \quad m > 1 \\ 0 & \text{if} \quad m = 1 \end{cases}. \tag{3.4}$$

If $f = \frac{\Theta_A}{1-u}$ then

$$\widehat{(\varphi_\sharp f)}(\iota_m) = \begin{cases} \widehat{\Theta}_A(\chi)\Big(1 - \chi(T)\Big)^{-1} & \text{if} \quad \chi(T) \neq 1 \\ \\ 0 & \text{if} \quad \chi(T) = 1 \end{cases}. \tag{3.5}$$

If $\varphi : A \to B$ is a surjective morphism of finite Abelian groups then we get morphisms

$$\varphi_\sharp : \mathbb{Q}[A]_0 \to \mathbb{Q}[B]_0, \quad \varphi^\sharp : B^\sharp \to A^\sharp.$$

Then for $f \in \mathbb{Q}[A]_0$ and $\chi \in B^\sharp$ we have

$$\widehat{(\varphi_\sharp f)}(\chi) = \begin{cases} \hat{f}(\varphi^\sharp \chi) & \text{if} \quad \chi \neq 1 \\ 0 & \text{if} \quad \chi = 1 \end{cases}.$$

We can now return to topology. We will continue to use the notations in the previous section. Applying [22, Lemma 6.2] iteratively we deduce the following result.

Theorem 3.3. *Suppose χ is a nontrivial character of $G = H_1(N, \partial N; \mathbb{Z})$, so that $\chi(G) = U_m$, for some $m > 1$. Then*

$$\chi_\sharp \mathcal{T}_{p/q} = p\chi_\sharp \mathcal{T}_{1/0} + q\chi_\sharp \mathcal{T}_{0/1}.$$

Above and in the sequel we use the convention $Object_{p/q} := Object(M_{p/q})$. To proceed further we take the Fourier transform of the above formula and we get

$$(\widehat{\chi_\sharp \mathcal{T}_{p/q}})(\iota_m) = (\widehat{p\chi_\sharp \mathcal{T}_{1/0}})(\iota_m) + (\widehat{q\chi_\sharp \mathcal{T}_{0/1}})(\iota_m),$$

where $m = \operatorname{ord}(\chi)$. Recall that

$$\mathcal{T}_{p/q} = \mathcal{T}^0_{p,q} - \frac{1}{2} CW_{p/q}\Theta_{p/q}, \quad \mathcal{T}_{1/0} = \mathcal{T}^0_{1/0} - \frac{1}{2} CW_{1/0}\Theta_{1/0},$$

$$\mathcal{T}_{0/1} = \mathcal{T}^0_{0/1} + \frac{\Theta_{0/1} T}{(1-T)^2}.$$

Using (3.4) and (3.5) and the identities $\Theta_{p/q}(\chi) = 0 = \Theta_{1/0}(\chi)$ for $\chi \neq 1$ we deduce we deduce

$$\widehat{\mathcal{T}}^0_{p/q}(\chi) = p\widehat{\mathcal{T}}^0_{1/0}(\chi) + q\widehat{\mathcal{T}}^0_{0/1}(\chi) + \begin{cases} \widehat{\Theta}_{0/1}(\chi)\chi(T)\Big(1 - \chi(T)\Big)^{-2} & \text{if} \quad \chi(T) \neq 1 \\ \\ 0 & \text{if} \quad \chi(T) = 1 \end{cases}.$$

The last term is nontrivial if and only if $\chi(T) \neq 1$ and $\chi\, |_{T(H_{0/1}} = 1$. This is possible if and only if $\chi = \chi_0^j$, for some $j = 1, 2, \cdots, m_0 - 1$. Additionally $\Theta_{0/1}(\chi_0^j) = |T(H_{0/1}| = |G|/m_0$. We conclude that if χ is a nontrivial character of G we have

$$\widehat{\mathcal{T}}^0_{p/q}(\chi) = p\widehat{\mathcal{T}}^0_{1/0}(\chi) + q\widehat{\mathcal{T}}^0_{0/1}(\chi) + \frac{|G|}{m_0}\begin{cases} \frac{\chi_0^j}{(1-\chi_0^j)^2} & \text{if} \quad \chi = \chi_0^j, \;\; j = 1, \cdots, m_0 - 1 \\ & \\ 0 & \text{if} \quad \chi \neq \chi_0^k, \;\; k = 1, \cdots, m_0 \end{cases}$$

We need to relate

$$\widehat{\mathcal{T}}^0_{p/q}(1) = \frac{|H_{p/q}|}{2} CW_{p/q} = \frac{pm_0|G|}{2} CW_{p/q},$$

$$\widehat{\mathcal{T}}^0_{1/0}(1) = \frac{|H_{1/0}|}{2} CW_{1/0} = \frac{m_0|G|}{2} CW_{1/0},$$

and

$$\widehat{\mathcal{T}}^0_{0/1}(1) = \frac{1}{2}\Delta''_{M_{0/1}}(1).$$

This follows from the surgery formula for the Casson-Walker invariant [7, §4.6], [23, Chap.4]. More precisely, the arguments in [15, p.38-39] yield

$$CW_{p/q} = pCW_{1/0} + \frac{q}{2}\Delta''_{M_{0/1}}(1) + |G|\left(\frac{q(m_0^2 - 1)}{12m_0} - \frac{pm_0 s(q,p)}{2}\right),$$

where $s(q,p)$ denotes the Dedekind sum of the pair of co-prime integers q, p. Putting all of the above together we deduce that for every pair of coprime integers (p,q) and every positive integer m_0 there exists a function $G_{p,q,m_0} : \mathbb{Z}/m_0\mathbb{Z} \to \mathbb{C}$ such that

$$\widehat{\mathcal{T}}^0_{p/q}(\chi) = p\widehat{\mathcal{T}}^0_{1/0}(\chi) + q\widehat{\mathcal{T}}^0_{0/1}(\chi) + |G|\begin{cases} G_{p,q,m_0}(j) & \text{if} \quad \chi = \chi_0^j \\ & \\ 0 & \text{otherwise} \end{cases}. \tag{3.6}$$

The similarity with (3.1) is striking. The results in [10, 15] show that

$$F_{p,q,m_0}(1) = G_{p,q,m_0}(1), \quad \forall p, q, m_0.$$

In particular

$$F_{p,q,1} = G_{p,q,1}, \;\; \forall p, q.$$

Let us briefly comment on the "flavor" of the surgery formulæ (3.1) and (3.6). Note first that the first homology group of a rational homology 3-sphere can be naturally identified with its dual using the linking form. We can think of the invariants $\mathcal{T}^0_M$ and $\mathbf{SW}^0_M$ as functions on $H^1(M, \mathbb{Z})$, as well as functions on the dual.

Suppose we perform Dehn surgery on a knot $K \hookrightarrow M$ to obtain a new rational homology sphere $M(K)$. The surgery formula essentially states that if we know the values of these

invariants on homology classes $c \in H_1(M,\mathbb{Z})$ which do not link with K then we can also compute the values of these invariants on homology classes $c \in H_1(M(K),\mathbb{Z})$ which do not link with $K \hookrightarrow M(K)$.

More rigorously, consider a pair M_0, M_1 related by a Dehn surgery on a knot K. Denote by N the common knot complement, and set

$$H := H_1(N,\mathbb{Z}), \;\; G := H_1(N,\partial N;\mathbb{Z}), \;\; H_i := H_1(M_i,\mathbb{Z}), \;\; i = 0,1.$$

We have a diagram of surjective morphisms

$$\begin{tikzcd} & H \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ H_0 & & H_1 \end{tikzcd}$$

Dualizing we get the diagram

$$\begin{tikzcd} & H^\sharp & \\ H_0^\sharp \arrow[ur, hook, "\pi_0^\sharp"] & & H_1^\sharp \arrow[ul, hook', "\pi_1^\sharp"'] \end{tikzcd}$$

The group G can be identified with a subgroup of $H^\sharp$. The knot K defines two subgroups

$$K_i^\perp := \Big\{ c \in H_i^\sharp;\;\; c(K_i) = 1 \Big\} = \Big\{ c \in H_i^\sharp;\;\; \mathbf{lk}_{M_i}(c, K_i) = 0 \Big\}, \;\; i = 0,1,$$

and we have isomorphisms $\pi_i^\sharp : K_i^\perp \to G$. We can think of G as the graph of a correspondence $T_K : H_0^\sharp \to H_1^\sharp$ induced by the Dehn surgery. The domain of this correspondence is $K_0^\perp$, the range is $K_1^\perp$, and viewed as a correspondence $T_K \subset K_0^\perp \times K_1^\perp$ it is a group isomorphism. We will refer to a such a correspondence as a *partial isomorphism* (p.i.) of groups.

For a surgery along a knot $K \hookrightarrow M$, whose meridian satisfies $\lambda_0 \cdot \mu = 1$, and attaching curve $c = p\mu + q\lambda_0$, we will denote by $\xi := \xi_{K,c}$ the induced p.i. We denote by G_K the group $H_1(M \setminus K, \partial(M \setminus K);\mathbb{Z})$, by $m_0(K,c)$ respectively $p = p(K,c)$ the divisibility, and respectively multiplicity of the surgery. Finally, set $D_M := \mathbf{SW}_M^0 - \mathcal{T}_M^0$. Since $\mathcal{T}_M^0 = \mathbf{SW}_M^0$ if $b_1(M) > 0$ we can now rephrase the surgery formulæ (3.1) and (3.6)

$$\widehat{D}_{M_{k,c}}(\xi_{K,c}\chi) = p\widehat{D}_M + |G_K|\mathcal{K}_{p,q,m_0}, \;\; \forall \chi \in K^\perp = \mathrm{Dom}\,(\xi_{K,c}),$$

where the correction term is a function on G_K, which is nontrivial only on the cyclic group of order m_0 generated by $\mathbf{j}\lambda_0 \in T(H_1(N \setminus K,\mathbb{Z})) = G_K^\sharp$, and depends only on the arithmetic of the surgery, p, q, m_0. Moreover $\mathcal{K}_{p,q,1} = 0$.

More generally consider a 3-manifold N such that $\partial N \cong T^2$, fix a longitude λ_0, and two primitive classes c_0, c_1 represented by two simple closed curves. By Dehn surgery with attaching curves c_0, c_1 we get two manifolds M_{c_0}, M_{c_1}, with first homology groups H_{c_0}, H_{c_1}, and distinguished classes $K_{c_i} \in H_{c_i}$, $i = 0,1$, defined by the core of the attached solid torus. Set $G := H_1(N,\partial N;\mathbb{Z})$, and denote by ξ_{c_1,c_0} the isomorphism $K_{c_0}^\perp \to K_{c_1}^\perp$ induced

by the surgery cobordism. We denote by $[c_0, c_1]$ the orbit of (c_0, c_1) relative to the action of $SL_2(\mathbb{Z})$ on the space of pairs of primitive classes $c_0, c_1 \in H_1(\partial N, \mathbb{Z})$. Then we have

$$(\lambda_0 \cdot c_0)\widehat{D}_{M_{c_1}}(\xi_{c_0,c_1}\chi) = (\lambda_0 \cdot c_1)\widehat{D}_{M_{c_0}}(\chi) + |G|\mathcal{K}_{[c_0,c_1],m_0}(\chi), \ \ \forall \chi \in K_{c_0}^{\perp}. \tag{3.7}$$

The **arithmetic type** α of a surgery is the pair $([c_1, c_2], m_0)$. We denote by $\mathcal{A}$ the set of all arithmetic types for which the correction term $\mathcal{K}$ is trivial. We know that

$$([c_1, c_2], 1) \in \mathcal{A}, \ \ \forall c_1, c_2.$$

We will call the surgeries of arithmetic type $\alpha \in \mathcal{A}$ as **admissible**.

Remark 3.4. As explained in [13, Remark B.6], the orbit $[c_0, c_0]$ is completely characterized by the extension

$$0 \to \mathbb{Z}\langle c_0\rangle \oplus \mathbb{Z}\langle c_1\rangle \hookrightarrow H^1(\partial N, \mathbb{Z}) \to H^1(\partial N, \mathbb{Z})/\Big(\mathbb{Z}\langle c_0\rangle \oplus \mathbb{Z}\langle c_1\rangle\Big) \to 0.$$

More precisely, the quotient group $H^1(\partial N, \mathbb{Z})/(\mathbb{Z}\langle c_0\rangle \oplus \mathbb{Z}\langle c_1\rangle)$ is isomorphic to the cyclic groups of order $|c_0 \cdot c_1|$, and the extension is characterized by a character of this group. Thus, the orbit $[c_0, c_1]$ is described by the integer $c_0 \cdot c_1$, and a character of $\mathbb{Z}_{|c_0 \cdot c_1|}$.

4 Seiberg-Witten $\Longleftrightarrow$ Casson-Walker+ Reidemeister torsion

§4.1 Topological preliminaries Denote by $\mathfrak{X}$ the family of all closed, compact oriented 3-manifolds M such that

$$\mathbf{SW}_M^0 = \mathcal{T}_M^0.$$

We want to prove that $\mathfrak{X}$ consists of all 3-manifolds.

We already know that $M \in X$ if $b_1(M) > 0$, or M is an *integral* homology sphere, or if M is a lens space. Also, we have

$$M_1, M_2 \in \mathfrak{X} \Longrightarrow M_1 \# M_2 \in \mathfrak{X}.$$

Definition 4.1. *A* **deflating primitive surgery** *is a Dehn surgery on a knot K in a rational homology sphere M with the following properties.*
(a) The longitude $\lambda \in H_1(\partial M \setminus K, \mathbb{Z})$ is a primitive class.
(b) The attaching curve c of the surgery satisfies $c \cdot \lambda = \pm 1$.

An **excellent** *surgery is a deflating primitive surgery which does not change the order of the first homology group. Two rational homology 3-spheres will be called* **e-related** *if one can be obtained from the other by a sequence of excellent surgeries.*

The attribute deflating is justified by the inequality

$$|H_1(M', \mathbb{Z})| \leq |H_1(M, \mathbb{Z})|$$

when M' is obtained from M by a deflating primitive surgery. The surgery is excellent iff we have equality. Note that if M_0 and M_1 are e-related then $H_1(M_0, \mathbb{Z}) \cong H_1(M_1, \mathbb{Z})$ and they have isomorphic linking forms. The following result is immediate.

Lemma 4.2. *Suppose $M \in \mathfrak{X}$, $b_1(M) = 0$. If M' is obtained from M by a deflating primitive surgery then $M' \in \mathfrak{X}$. In particular, if $M \in \mathfrak{X}$ and M' is e-related to M then $M' \in \mathfrak{X}$.*

Proof Indeed, we have $G := H_1(M', \mathbb{Z}) \cong H_1(M \setminus K, \partial(M \setminus K); \mathbb{Z})$ and $F_{p,q,1} = G_{p,q,1}$. The surgery formulæ establish the equality of $\mathcal{T}^0_{M'}$ and $\mathbf{SW}^0_{M'}$ as functions on $G^\sharp$, and $G^\sharp$ turns out to be their maximal domain. ■

Corollary 4.3. *$\mathfrak{X}$ contains lens spaces, integral surgery spheres, and is closed under connected sums and deflating primitive surgeries.*

Before we proceed further we want to briefly recall some basic topological facts. For more details we refer to [3, 16]. Any rational homology sphere can be obtained from S^3 by performing Dehn surgery on a link $L = K_1 \cup \cdots \cup K_n$ with surgery coefficients $p_1/q_1, \cdots, p_n/q_n$. Set $\vec{p} = (p_1, \cdots, p_n)$, $\vec{q} = (q_1, \cdots, q_n)$. We denote by $M = M(L, \vec{p}, \vec{q})$ the three manifold obtained by this surgery. We say that a surgery diagram belongs to $\mathfrak{X}$ if the corresponding 3-manifold belongs to $\mathfrak{X}$.

All the homological data of $M(L, \vec{p}, \vec{q})$ is contained in the $n \times n$ linking matrix $A = A(L; \vec{p}, \vec{q})$ defined by

$$a_{ij} = \begin{cases} p_i/q_i & \text{if} \quad i = j \\ \ell_{ij} & \text{if} \quad i \neq j \end{cases},$$

where $\ell_{ij} = \mathbf{Lk}(K_i, K_j)$. The surgery diagram is called integral if the linking matrix is integral.

Denote by μ_i the meridian of K_i, set $\Omega := A^{-1}$, and denote by Q the diagonal $n \times n$ matrix $Q := \operatorname{diag}(q_1, \cdots, q_n)$. The manifold M is a rational homology sphere if and only if the linking matrix A is nonsingular. The first homology group admits the presentation

$$0 \to \mathbb{Z}^n \xrightarrow{QA} \mathbb{Z}^n \to H_1(M, \mathbb{Z}) \to 0$$

so that its order is $\det(QA)$. Moreover the images of the knots K_i generate $H_1(M, \mathbb{Z})$ and we have

$$\mathbf{lk}_M(K_j, K_i) = -\Omega_{ji} \mod \mathbb{Z}.$$

We have a natural isomorphism $H_1(S^3 \setminus L; \mathbb{Z}) \to \mathbb{Z}^n$ defined by

$$c \mapsto \Big(\mathbf{Lk}(c, K_1), \cdots, \mathbf{Lk}(c, K_n)\Big)$$

for any closed curve disjoint from L. More geometrically

$$c = \sum_{i=1}^{m} \mathbf{Lk}(K_i, c)\mu_i.$$

Such a closed curve defines a homology class $[c]$. We have $[\mu_i] = -q_i[K_i]$.

Suppose $[c]$ is a homology class in M of order m. (We set $m = 1$ if $[c] = 0$.) A surgery on a knot representing $[c]$ has divisibility m_0 determined by

$$m_0 := (k, m), \quad \mathbf{lk}_M([c], [c]) = \frac{k}{m} \mod \mathbb{Z}.$$

A class $c \in H_1(M, \mathbb{Z})$ is called *primitive* if it has divisibility one. Note that if K is a node in M representing a primitive class of order m, and M' is a manifold obtained from M by deflating surgery then

$$|H_1(M', \mathbb{Z})| = \frac{1}{m}|H_1(M, \mathbb{Z})|.$$

The above observations show that the excellent surgeries are precisely the $1/q$-surgeries on a homologically trivial knots.

The **pruning** of a surgery diagram is the operation of removing the components with surgery coefficients ± 1 which are algebraically split from the rest of the diagram. The pruning is equivalent to performing a sequence of excellent surgeries. We say that two surgery diagrams are **p-related** if one can go form one to another by a sequence of Kirby moves and prunings.

Corollary 4.4. *If $\mathcal{D}$ is a surgery diagram p-related to a diagram in $\mathfrak{X}$ then $\mathcal{D}$ is also in $\mathfrak{X}$.*

For every $\vec{a} \in \mathbb{Z}^n$ we denote by $[\vec{a}]$ the rational number

$$[\vec{a}] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots}}.$$

Lemma 4.5. **[17, N. Saveliev]** *Any homology lens space is e-related to a lens space.*

Proof Any homology lens space M is obtained by Dehn surgery on a knot K_0 in an integral homology sphere M', [1]. Denote by $r \in \mathbb{Q}$ the surgery coefficient of K_0. We can represent the homology sphere M' as surgery on an algebraically split link $L = K_1 \cup \cdots \cup K_n$ in S^3 with surgery coefficients $\varepsilon_j = \pm 1$. We can think of M as obtained from S^3 by surgery on the link $L_0 = K_0 \cup L$. Suppose

$$r = [\vec{a}], \;\; \vec{a} = (a_1, a_2, \cdots, a_m).$$

Performing a sequence of slam-dunks as in [3, Sec. 5.3] we can replace L_0 with the link $L \cup K$ as in Figure 1. We have thus succeeded in presenting M as an integral surgery on a link in S^3 with linking matrix

$$\begin{bmatrix}
\pm 1 & 0 & 0 & \cdots & 0 & \ell_1 & 0 & \cdots & \cdots & 0 \\
0 & \pm 1 & 0 & \cdots & 0 & \ell_2 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \pm 1 & \ell_n & 0 & \cdots & \cdots & 0 \\
 & & & & & & & & & \\
\ell_1 & \ell_2 & \cdots & \cdots & \ell_n & a_1 & 1 & 0 & 0 & \cdots \\
0 & 0 & \cdots & \cdots & 0 & 1 & a_2 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & a_3 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 1 & a_m
\end{bmatrix}$$

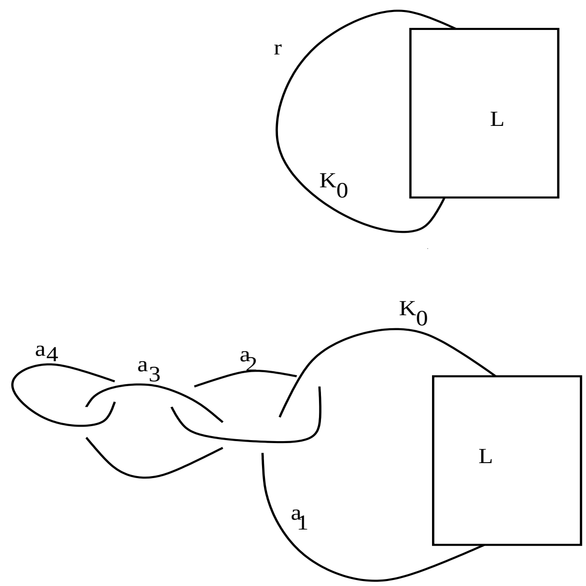

Figure 1: *Slam-dunking* K_0.

The first part of this matrix is described by the link L, and $\ell_j := \mathbf{Lk}(K_0, K_j)$. By sliding K_0 over the components of L we can kill the off-diagonal terms ℓ_i. This changes the topological type of K_0. It becomes a knot K_0', and the surgery coefficient a_1 changes to some integer a_1'. Reverse the slam-dunks. We get a new link algebraically split link $L_2 = K_0' \cup L$ where the surgery coefficient of K_0' is $r' = [a_1', a_2, \cdots, a_m]$, and the surgery coefficient of K_j is ± 1. By inserting ∞-unknots and performing a sequence of Rolfsen twists we can replace K_0' with an unknot K_0''. We can thus describe M as surgery on the algebraically split link

$$L_2 = K_0'' \cup K_1 \cdots \cup K_n$$

with surgery coefficients $\varepsilon_0 = r'$, $\varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1$. The r' surgery on K_0'' is a lens space while the surgeries on K_j are excellent surgeries. This shows M is e-related to a lens space. ■

§4.2 Proof of the main result We will present a proof by induction over the "complexity" of a rational homology sphere. To define the notion of complexity we need to present a few algebraic facts about the linking forms of such manifolds. We follow the notations in [4].

For each prime $p > 1$, each $q \in \mathbb{Z}$, and each $k \geq 1$ such that $(p, q) = 1$ denote by $A_p^k(q)$ the linking form n the cyclic group $\mathbb{Z}/p^k$ defined by

$$\mathbf{g} \cdot \mathbf{g} = \frac{q}{p^k}$$

where g denotes the natural generator of this group. Also denote by E_0^k , $k \geq 1$ and E_1^k, $k \geq 2$, the linking forms on $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ defined by the matrices

$$E_0^k = \begin{bmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{bmatrix}, \quad E_1^k = \begin{bmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{bmatrix}.$$

When referring to $A_p^k(q)$, E_0^k , E_1^k we mean the corresponding groups equipped with these linking forms. Define the complexity of A_p^n to be $\kappa(A_p^n) = p^{k+1}$. Define the complexity of E_i^n, $i = 0, 1$ to be $\kappa(E_i^n) = 2^{2n+2}$.

A classical result of C.T.C. Wall [24] shows that every linking form $(G, \mathfrak{q})$ decomposes non-uniquely into an orthogonal sum of A's and E's. If $\mathfrak{q}$ is a linking form on a p-group, then we define its complexity to be the product of the complexities of its elementary constituents A and/or E in some orthogonal decomposition. The results in [4] show that this number is independent of the chosen orthogonal decomposition of $\mathfrak{q}$ in elementary parts A and E. We denote by $\kappa(\mathfrak{q})$ the complexity of $\mathfrak{q}$. For every $\mathbb{Q}HS$ M we denote by ν_M the order of $H_1(M, \mathbb{Z})$, by $\mathfrak{q}_M$ the linking form of M, and by κ_M the complexity of $\mathfrak{q}_M$. We have the following elementary result whose proof is left to the reader.

Lemma 4.6. *If M_1 and M_2 are two rational homology spheres such that $\nu_{M_1} | \nu_{M_2}$ and $\nu_{M_1} < \nu_{M_2}$ then $\kappa_{M_1} < \kappa_{M_2}$.*

We would like to present a few methods of reducing the complexity of a manifold. The primitive deflating surgeries provide one first example.

Definition 4.7. *Let K be a knot in a rational homology sphere M supporting a* **nontrivial** *homology class. The knot K is called* **good** *if $\mathfrak{q}_M(K, K) \neq 0$. Otherwise, it is called* **bad**.

Suppose K is a good knot in a rational homology sphere M. If r is order of K then

$$\mathfrak{q}(K, K) = \frac{m}{r}, \;\; 0 < m < r,$$

and the divisibility of any surgery on this knot is $m_0(K) := (m, r)$. Consider any surgery with attaching curve c satisfying $|c \cdot \lambda| = m_0$. This is a surgery of divisibility m_0 and of type $(p, q) = (m_0, *)$. We obtain a new rational homology sphere M' such that $\nu_M = \frac{r}{m_0} \nu_{M'}$. Lemma 4.6 shows that the complexity of M' is smaller than the complexity of M. We have thus proved the following result.

Corollary 4.8. *The complexity of a rational homology sphere can be reduced by performing surgeries on good knots. Moreover if the original manifold has no 2-torsion we can arrange so that the resulting manifold also has no 2-torsion.*

Certain surgeries on certain bad knots also do reduce the complexity. We have the following technical result whose proof is deferred to an Appendix.

Lemma 4.9. *Suppose M is a rational homology sphere with linking form $A_p^s(q_1) \oplus A_p^r(q_2)$ $s \geq r$, and K is a bad knot in M of the form*

$$K = c_1 \oplus c_2$$

where the homology class c_2 generates $A_{p^r}(q_2)$. Then one can perform a surgery on K such that the resulting manifold is a homology lens space of the same order as M.

We will call the surgery in this lemma A_p-surgery. A knot with the properties in the lemma will be called a **mildly bad** knot. Set

$$\mathcal{Q} := \Big\{ \mathfrak{q}; \;\; \mathfrak{q}_M \cong \mathfrak{q} \Longrightarrow M \in \mathfrak{X} \Big\}.$$

We already know that all the linking forms $A_p^k(q)$ belong to $\mathcal{Q}$.

We need to talk a little bit about admissible surgeries, i.e. surgeries for which the correction term in the surgery formula (3.7) is trivial. Observe that if two rational homology spheres in $\mathfrak{X}$ are related by a Dehn surgery then this surgery is admissible.

Corollary 4.10. *The surgeries A_p surgeries described in Lemma 4.9 are admissible.*

Proof Consider a direct sum of two lens spaces with the above linking forms. This is a manifold in $\mathfrak{X}$. The result of this surgery produces a rational homology space which is also a manifold in $\mathfrak{X}$ so the surgery is admissible. ■

We also want to mention the following topological result. For a proof we refer to [13].

Lemma 4.11. *Suppose M_1, M_2 are two rational homology spheres and ϕ is an isomorphism*

$$\phi : (H_1(M_1,\mathbb{Z}), \mathfrak{q}_{M_1}) \to (H_1(M_2,\mathbb{Z}), \mathfrak{q}_{M_2}).$$

Suppose K_i is a knot in M_i, $i = 1, 2$ such that $\phi([K_1]) = [K_2]$. If M_i' $i = 1, 2$, are obtained perform surgeries **of the same arithmetic type** *α on K_1 and K_2, then there exists an isomorphism*

$$\phi_\alpha : (H_1(M_1',\mathbb{Z}), \mathfrak{q}_{M_1'}) \to (H_1(M_2',\mathbb{Z}), \mathfrak{q}_{M_2'}).$$

The main trick used in the proof is the following immediate consequence of the surgery formula (3.7).

Lemma 4.12. *Suppose M is a rational homology sphere, and χ a character of $H = H_1(M,\mathbb{Z})$. We identify H with its dual using the linking form. Suppose that there exists an admissible surgery on a knot K_χ such that*

$$\mathfrak{q}_M(\chi, K) = 0 \Longleftrightarrow \chi \in K^\perp$$

and the result of the surgery is a manifold in $\mathfrak{X}$. Then $\widehat{D}_M(\chi) = 0$. In particular, if for every χ there exists a knot with the above properties then $\widehat{D}_M \equiv 0$.

The proof of Theorem 2.4 will be carried out in several steps.

Step 1 Fix a prime number $p > 2$, and denote by $\mathcal{R}_p$ the family of rational homology spheres such that$\nu_M = p^r$, $r > 0$. We will show that $\mathcal{R}_p \subset \mathfrak{X}$. The proof will be an induction on the complexity. For $\kappa \geq 0$ denote by $\mathcal{R}^\kappa$ the manifolds in $\mathcal{R}_p$ of complexity $\leq \kappa$.

The rational homology lens spaces of order p have minimal nonzero complexity $p + 1$, and belong to $\mathfrak{X}$ so that

$$\mathcal{R}_p^{p+1} \subset \mathfrak{X}.$$

Observe that if $M \in \mathcal{R}_p$ then we have a decomposition

$$\mathfrak{q}_M = \bigoplus_{j=1}^{n} A_p^{s_j}(q_j), \quad 0 < s_1 \leq s_2 \leq \cdots \leq s_k,$$

and

$$\kappa_M = p^{s_1 + \cdots s_k + k}.$$

The integer k is called the *rank*, and we denote it by ρ_M. Define the **the standard model** of M to be the connected sum of lens spaces with the same linking form as M. We denote the standard model by $\tilde{M}$. Note that for every $M \in \mathcal{R}_p$ we have $\tilde{M} \in \mathfrak{X}$.

Suppose $\mathcal{R}_p^\kappa \subset \mathfrak{X}$. We want to prove that $\mathcal{R}_p^{\kappa+1} \subset \mathfrak{X}$. Let $M \in \mathcal{R}_p^{\kappa+1}$. Set $H := H_1(M, \mathbb{Z})$, and fix a nontrivial character χ of H. We distinguish two cases.

Case 1 There exists a good knot $K \in \chi^\perp$. Then there exists a good knot $\tilde{K}$ in the model $\tilde{M}$. We can perform a complexity reducing surgery on $\tilde{K}$ to obtain a manifold of smaller complexity which by induction we know is in $\mathfrak{X}$. This show that the arithmetic type of this surgery is admissible. We perform this admissible surgery on the knot K on M and we obtain a manifold of smaller complexity. Lemma 4.12 then implies $\widehat{D}_M(\chi) = 0$.

Case 2 $\chi^\perp$ consists only of bad knots. If the rank of H is 1 then M is a rational homology space so it is in $\mathfrak{X}$. Suppose the rank is > 1. $(H, \mathfrak{q}_M)$ decomposes into a nontrivial sum of cyclic p groups

$$H = \mathbb{Z}/p^{s^1} \oplus \cdots \oplus \mathbb{Z}/p^{s^k}, \;\; 0 < s_1 \leq \cdots \leq s_k, \;\; k > 1.$$

We get a corresponding decomposition $\chi = \chi_1 \oplus \cdots \oplus \chi_k$. Observe that all components must be nonzero. Indeed, if $\chi_j = 0$ then the generator of the j-th component belongs to $\chi^\perp$ and is a good knot. Thus $\chi_1, \chi_2 \neq 0$. It is easy to see that $\chi^\perp \cap \mathbb{Z}/p^{s^1} \oplus \mathbb{Z}/p^{s_2} \neq 0$. Pick a mildly bad knot K in this group. Thus all but the first two components of K are zero, and one of the components generates the corresponding summand. Perform an A_p surgery on this knot. This reduced the complexity of M. By induction, the resulting manifold is in $\mathfrak{X}$ that $\hat{D}_M(\chi) = 0$. Thus $\mathcal{R}_p \subset \mathfrak{X}$

Step 2 If ν_M is odd then $M \in \mathfrak{X}$. For each vector $\vec{p} = (p_1, \cdots, p_n)$ whose components consist of pairwise distinct of odd primes. Denote by $\mathcal{R}_{\vec{p}}$ the family of rational homology spheres M such that the prime divisors of ν_M are amongst the primes p_j. Again we perform induction on complexity. The first homology group H of $M \in \mathcal{R}_{\vec{p}}$ decomposes as an orthogonal direct sum of p-groups

$$H = \bigoplus_{j=1}^n G_{p_j}, \;\; |G_{p_j}| = p_j^{s_j}.$$

Each component G_{p_j} decomposes as an orthogonal sums of $A^*_{p_j}(*)$. Denote by r_j the number of such components. We have

$$\kappa(M) = |H| + r_1 + \cdots + r_n.$$

Suppose χ is a nontrivial character of H. We distinguish again two cases.

Case 1 $\chi^\perp$ contains good knots. In this case we perform a surgery as in Lemma 4.8 which produces a manifold of smaller complexity. Using models as in **Step 1** we can prove that such a surgery is admissible. Thus in this case $\hat{D}_M = 0$.

Case 2 If all r_j's are $= 1$ then H is a cyclic group, M is a homology space, so that $M \in \mathfrak{X}$.

Suppose $r_1 > 1$. We set $p := p_1$ and

$$G_p = \bigoplus_{j=1}^{n} A_p^{s_j}(q_j), \quad 0 < s_1 \leq s_2 \leq \cdots \leq s_k.$$

We conclude as in **Step 1** that all the components of χ determined by the above decomposition of G_p are nontrivial. By performing an A_p surgery on a mildly bad knot we obtain a manifold M' satisfying

$$\nu_{M'} = \nu_M, \quad r_1' = r_1 - 1, \quad r_j' = r_j, \quad \forall j = 2, \cdots n.$$

Thus $\kappa(M') < \kappa(M)$ and we conclude by induction. We can now conclude that any Dehn surgery which transforms an odd order $\mathbb{Q}HS$ to an odd order $\mathbb{Q}HS$ is admissible. We define the complexity of an odd order $\mathbb{Q}HS$ to be the product of the complexities of the p-groups it decomposes into.

Step 3 If $\mathfrak{q}_M = A_2^n(q) \oplus \mathfrak{q}_1$, where $\mathfrak{q}_1$ is a linking form on a group of odd order, then $M \in \mathfrak{X}$. Denote by $\mathcal{R}_2'$ the family of such rational homology spheres. Define the complexity of such a manifold to be

$$\kappa_M = 2^{n+1}\kappa(\mathfrak{q}_1).$$

The considerations in **Step 2** lead to the following complexity reduction trick.

Lemma 4.13. *If K is a knot is an odd order $\mathbb{Q}HS$ such that $K^\perp$ is a nontrivial subgroup, then there exists $K' \in K^\perp$ and a surgery on K' producing an odd order $\mathbb{Q}HS$ of smaller complexity. Moreover, such a surgery is admissible.*

Suppose $\mathfrak{q}_M = A_2^n(q) \oplus \mathfrak{q}_1$ and χ is a nontrivial character of $\mathfrak{q}_M$. Then $\chi^\perp \neq 0$. Decompose

$$\chi = \chi_0 \oplus \chi_1, \quad \chi_0 \in A_2^n(q), \quad \chi_1 \in \mathfrak{q}_1$$

It follows that $\chi_1^\perp$ is a nontrivial subgroup in $\mathfrak{q}_1$. Perform a complexity reduction surgery on a knot $K \in \chi_1^\perp \subset \mathfrak{q}_1$ as in Lemma 4.13 to conclude as we have done before that $\widehat{D}_M(\chi) = 0$.

Step 4 If $\mathfrak{q}_M = \bigoplus_{k=1}^m A_2^{n_k}(q_k) \oplus \mathfrak{q}_1$, $n_1 \geq n_2 \geq \cdots \geq n_m > 0$, where $\mathfrak{q}_1$ is a linking form on a group of odd order, then $M \in \mathfrak{X}$. Denote by $\mathcal{R}_2$ the family of such rational homology spheres. Define the complexity of such a manifold to be

$$\hat{\kappa}_M := \kappa\Big(\bigoplus_{k=1}^m A_2^{n_k}(q_k)\Big) = 2^{n_1+\cdots+n_m+m}.$$

For every $M \in \mathcal{R}_2$ we define its model $\tilde{M}$ to be a connected sum of lens spaces with the same linking form as M. Again we will carry a proof by induction on the complexity. The basic complexity reduction technique is contained in the following lemma whose proof is deferred to the Appendix.

Lemma 4.14. *(a) Suppose $c \in A_2^s(q_1) \oplus A_2^r(q_2)$, $s \geq r > 0$, . Then there exists $K \in c^\perp$ of the form*

$$K = K_1 \oplus K_2 \tag{4.1}$$

where K_2 is a generator of $A_2^r(q_2)$.
(b) Suppose M is a rational homology sphere such that $\mathfrak{q}_M = A_2^s(q_1) \oplus A_2^r(q_2)$ and K is a knot in M whose homology class satisfies (4.1). Then there exists a Dehn surgery on M such that the resulting manifold M' is in $\mathcal{R}_2'$ and has smaller complexity. More precisely, we can arrange so that

$$\mathfrak{q}_{M'} = A_2^t(q) \oplus \mathfrak{q}_1$$

where $t \leq r + s$ and $\mathfrak{q}_1$ is the linking form of some odd order lens space.

Let $M \in \mathcal{R}_2$. Then we can write

$$\mathfrak{q}_M = \mathfrak{q}_0 \oplus \mathfrak{q}_1 := (\bigoplus_{k=1}^m A_2^{n_k}(q_k) \oplus \mathfrak{q}_1.$$

If $m = 1$ then $M \in \mathfrak{X}$ according to **Step 3**. We can assume $m > 1$.

Any nontrivial character $\chi \in \mathfrak{q}_M$ decomposes as

$$\chi = \chi_0 + \chi_1, \;\; \chi_i \in \mathfrak{q}_i, \;\; i = 0, 1.$$

Pick $K \in A_2^{n_1}(q_1) \oplus A_2^{n_2}(q_2)$ orthogonal to χ_0, and satisfying (4.1). We want to perform a complexity reduction surgery as in Lemma 4.14 but we first must show that any such surgery is admissible. This can be seen by performing this surgery on the model $\tilde{M} \in \mathfrak{X}$. It produces a manifold of smaller complexity which by induction we know it is in $\mathfrak{X}$, and thus proving that the surgery is admissible. We can now conclude as many times before that $\widehat{D}_M(\chi) = 0$. This shows that $\mathfrak{R}_2 \subset \mathfrak{X}$.

Step 5 Conclusion Suppose M is an arbitrary $\mathbb{Q}HS$. Then $\mathfrak{q}_M = \mathfrak{q}_0 \oplus \mathfrak{q}_1$, where $\mathfrak{q}_0$ is a linking form on a 2 group, and $\mathfrak{q}_1$ is a linking form on an odd order group. The results in [4, Theorem 0.1] show that if we add sufficiently many A's to $\mathfrak{q}_0$ we obtain a linking form isomorphic to a direct sum of A's. Topologically this means that we can find a connected sum X of lens spaces of order 2^s such that $M\#X \in \mathcal{R}_2 \subset \mathfrak{X}$. Thus $\widehat{D}_{M\#X} = 0$. Since $\widehat{D}$ is additive with respect to connected sums we deduce $\widehat{D}_M = 0$. This concludes the proof of Theorem 2.4. ■.

5 Final comments

The invariant introduced by Ozsváth and Szabó in [15] satisfies the same surgery formula as the modified Seiberg-Witten invariant, and detects in the same fashion the Casson-Walker invariant. This shows that the strategy presented in this paper also answers a question in [15]. More precisely, their invariant is equivalent to the modified Reidemeister torsion.

If we consider the mod $\mathbb{Z}$ reduction of the modified Seiberg-Witten invariant we deduce that

$$\mathbf{sw}_M^0(\sigma) = \frac{1}{8} KS_M(\sigma) \mod \mathbb{Z},$$

where $KS_M(\sigma)$ denotes the Kreck-Stolz invariant. It general it depends on the metric but its mod $8\mathbb{Z}$ reduction is metric independent. Fix a *spin* structure ϵ. This choice allows us to think of $\mathcal{T}$ and $\mathbf{SW}$ as functions $H \to \mathbb{Q}$, $H := H_1(M, \mathbb{Z})$.

Denote by $\mathcal{F}_M$ the space of functions $f : H \to \mathbb{Q}/\mathbb{Z}$. For each $h \in H$ define the finite difference operator

$$\Delta_h : \mathcal{F} \to \mathcal{F}, \ \ (\Delta_h f)(\sigma) := f(h \cdot \sigma) - f(\sigma).$$

In [20] its is shown that for every $h_1, h_2 \in H$ we have

$$\mathbf{lk}_M(h_1, h_2) = \Delta_{h_1} \Delta_{h_2} \mathcal{T} \mod \mathbb{Z}.$$

Since the constant functions are killed by $\Delta_\bullet$ we deduce

$$\Delta_\bullet \mathcal{T} = \Delta_\bullet \mathcal{T}^0.$$

Our main result now implies the following equality.

$$\mathbf{lk}_M(h_1, h_2) = -\frac{1}{8} \Delta_{h_1} \Delta_{h_2} KS_M \mod \mathbb{Z}, \ \ \forall h_1, h_2 \in \mathbb{Z}.$$

It would be interesting to investigate whether this identity has a higher dimensional counterpart.

In [14] we associated to each *spin* structure ϵ on a rational homology sphere an invariant $c(\epsilon) \in \mathbb{Q}/\mathbb{Z}$ which was powerful enough to distinguish many lens spaces. We can now identify it. We have

$$c(\epsilon) = \frac{1}{8} KS_M(\epsilon) \mod \mathbb{Z}.$$

A Proofs of some technical results

Proof of Lemma 4.9. A simple model of A_p surgery is the manifold given by the surgery diagram

$$\mathcal{D}_n := \Big\{ (K_1, p^s/q_1), (K_2, p^r/q_2), (K, n) \Big\}, \ \ s \geq r > 0$$

where K_1 and K_2 are unlinked, unknots, and K is a knot such that $\mathbf{Lk}(K, K_i) = \ell_i$, $-p^s/2 < \ell_1 < p^s/2$, $-p^r/2 < \ell_2 < p^r/2$. The linking matrix of this diagram is

$$\begin{bmatrix} \frac{p^s}{q_1} & 0 & \ell_1 \\ 0 & \frac{p^r}{q_2} & \ell_2 \\ \ell_1 & \ell_2 & n \end{bmatrix}.$$

We can view K as a knot in the connected sum of lens spaces $L(p^s, -q_1)\#L(p^r, -q_2)$. K is a bad knot if and only if

$$\frac{q_1\ell_1^2}{p^s} + \frac{q_2\ell_2^2}{p^r} = k \in \mathbb{Z}. \qquad (*)$$

The knot is mildly bad if and only if $(p, \ell_2) = 1$. Denote by H the first homology group of the 3-manifold obtained by performing the surgery indicated by $\mathcal{D}_n$. The matrix

$$B_n := \begin{bmatrix} p^s & 0 & q_1\ell_1 \\ 0 & p^r & q_2\ell_2 \\ \ell_1 & \ell_2 & n \end{bmatrix}$$

is a presentation matrix for H and has determinant

$$\det(B_n) = p^{s+r}n - p^s q_2\ell^2 - p^r q_1\ell_1^2 = p^{s+r}\left(n - \left(\frac{q_1\ell_1^2}{p^s} + \frac{q_2\ell_2^2}{p^r}\right)\right) = p^{s+r}(n-k).$$

Observe that $|\det B_n| = p^{s+r}$ when $n = k \pm 1$. Let $n = k+1$.

Rewrite the condition $(*)$ as

$$q_1\ell_1^2 + p^{s-r}q_2\ell_2^2 = p^s k$$

Since $(q_1\ell_1, p) = 1$ we deduce that $(q_2\ell_2, p) = 1$. To find H we need to find the elementary divisors $d_1|d_2|d_3$ of B_{k+1}. Clearly $d_1 = 1$. By looking at the 2×2 minor in the top left hand corner we deduce that $d_2|p^{s+r}$. On the other hand, if we look the 2×2 minor

$$\begin{vmatrix} 0 & q_2\ell_2 \\ \ell_1 & k+1 \end{vmatrix}$$

we deduce that it is not divisible by p. Thus $d_2 = 1$ which shows that H is a cyclic p-group of order p^{s+r}. ■

Proof of Lemma 4.14. Part (a) is elementary and is left to the reader. For part (b) it suffices to look at a concrete realization of the given homological data. Any homology class $K \in A_2^s(q_1) \oplus A_2^r(q_2)$ satisfying (4.1) can be realized as a knot in a connected sum of lens spaces $X := L(2^s, a)\#L(2^r, b)$. We present X as two unlinked unknots K_1, K_2 with surgery coefficients $-2^s/a$, $-2^r/b$, and K as a knot such that

$$\mathbf{Lk}(K, \mathbf{g}) = 1, \;\; \mathbf{LK}(K, K_1) = k.$$

Assume K has an integral surgery coefficient n (see Figure 2). Slam-dunking K_2 over K we obtain a surgery presentation with linking matrix

$$A := \begin{bmatrix} -2^s/a & k \\ k & n + b/2^r \end{bmatrix}.$$

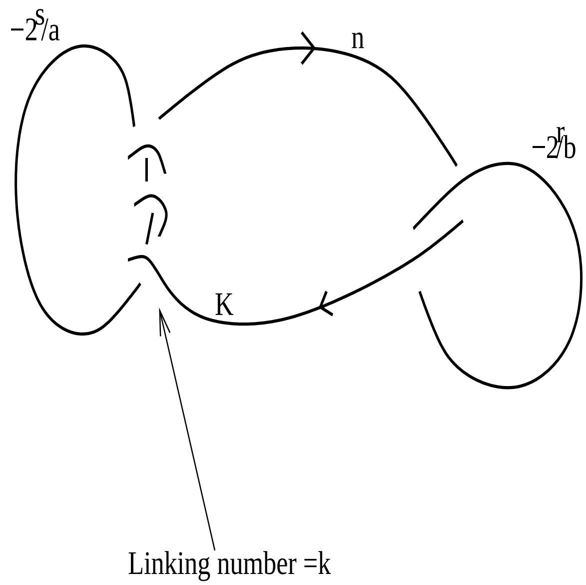

Figure 2: *Modeling a complexity reducing surgery*

The first homology group H of the manifold described by this surgery diagram admits the presentation matrix

$$B := \begin{bmatrix} -2^s & ak \\ 2^r k & 2^r n + b \end{bmatrix}.$$

The order of this group is $|2^{r+s}n + 2^s b + 2^r ak^2|$. Pick n to be any number such that 2^{r+s+1} does not divide the order of this group. Then H is a cyclic group of the form $\mathbb{Z}/2^t \oplus \mathbb{Z}/(2m+1)$, $t \leq r+s$. Its $\hat{\kappa}$-complexity is smaller than that of $A_2^s(q_1) \oplus A_2^r(q_2)$ $\blacksquare$

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